MST121 : Supplementary Resource Material for Chapter C3

Question 1

Find the general solution of the differential equation $\frac{dy}{dx} = x^2(1+x^3)^4$.

(i) Find the particular solution of this differential equation that satisfies $y(0) = 1$.

(ii) What is the value of y when $x = -1$?

Question 2

Solve the initial value problem $\frac{dx}{dt} = e^{-t}$, $x = 3$ when $t = \ln 2$.

Question 3

Verify that $y = 4e^{3x} - 3e^{4x} + xe^{4x}$ is a solution of the second order differential equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{4x}, \quad y(0) = y'(0) = 1$$

Question 4

Verify that $y^2 - 5y = x^3 + 8x$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2 + 8}{2y - 5}$$

and find the slope of the solution curve at the point where the curve cuts the x-axis.

Question 5

Use the method of separation of variables to solve each of the following first order differential equations:

(a) $\frac{2y}{3x^2}\frac{dy}{dx} = \frac{y^2+1}{x^3-1}$, $\quad y(3) = 1$, $\quad x \geq \sqrt[3]{14}$

(b) $\sec x \frac{dy}{dx} = e^{-y}$, $\quad y(\frac{\pi}{6}) = \ln 2$

(c) $(1+\cos 2x)\frac{dy}{dx} - e^y \sin 2x = 0$, $\quad y(\frac{\pi}{4}) = \ln 2$

(d) $\frac{1+y}{y}\frac{dy}{dx} - x = xy$, $\quad y(1) = 1$

(e) $y \sin x \frac{dy}{dx} = \cos x, \quad y(\tfrac{\pi}{6}) = 1$

(f) $r \frac{d\psi}{dr} = \cos^2 \psi, \quad r(\tfrac{\pi}{4}) = 1$

Question 6

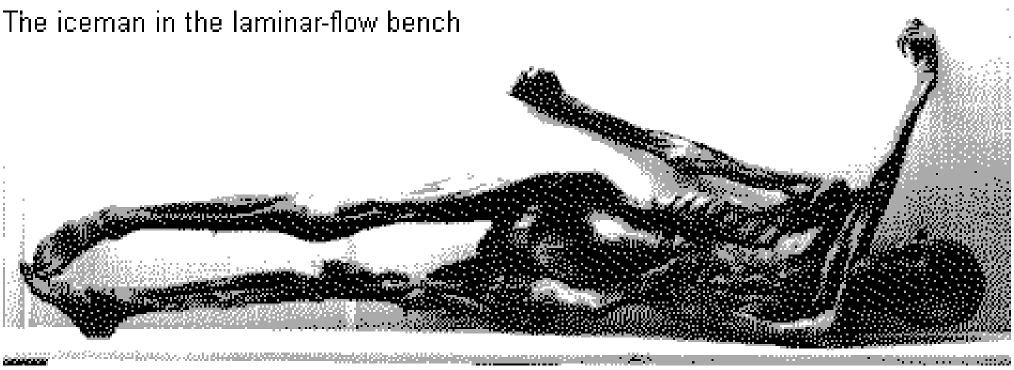
(a) Show that $\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$ can be re-written as $\frac{1}{x^2-1}$

(b) A curve that passes through the point with co-ordinates $(2,5)$ has the property that its gradient is given by $\frac{y+1}{x^2-1}$ at any point (x, y) on the curve. Use the result of part (a) to find the equation of the curve, giving your answer in the form $y = f(x)$ where f is a function to be found.

Question 7

The Iceman

The 'Iceman' is the oldest mummified human body ever found intact. It was discovered by a German tourist, Helmut Simon, on the Similaun Glacier in the Tirolean Ötztal Alps on the Italian-Austrian border, on Sept. 19, 1991. The body has been Radiocarbon-dated, and during this process it was established that the body contained approximately 53% of the amount of Carbon-14 contained in living tissue.



(a) Use the fact that the half life of carbon-14 is 5570 years to find the value of the decay constant, k, in the equation $m = m_0 e^{-kt}$.

(b) Now use the fact that the body contained almost 53% of the amount of Carbon-14 contained in living tissue to find an approximate age of the skeleton.

Question 8

(a) In a chemical reaction in which a compound X is formed from a compound Y and other substances, the masses of X and Y present at time t are x and y respectively. The sum of the two masses is constant and at any time the rate at which x is increasing is proportional to the product of the masses at that time. Show that the equation governing the reaction is of the form $\frac{dx}{dt} = kx(a - x)$, and interpret the constant a.

(b) Show that $\frac{1}{a}\left(\frac{1}{x} + \frac{1}{a - x}\right) = \frac{1}{x(a - x)}$.

(c) If $x = \frac{1}{10}a$ at time $t = 0$, find, in terms of k and a, the time at which $y = \frac{1}{10}a$.

Question 9

The table below shows lower bounds for the world's population for the years between 1900 and 1950.

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400

Source: http://www.census.gov/ipc/www/worldhis.html

(a) By using a log-linear plot for the data in this table, show that the exponential model $P = P_0 e^{Kt}$ is appropriate for the world's population for the years between 1900 and 1950. Measure time t in years, from $t = 0$ at the 1900 census date.

(b) Use your plot to estimate values for the proportionate growth rate K and the initial population size P_0 in the exponential model.

(c) Use the predictions from the model to complete the table below

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400
Model's prediction						
Percentage error						

(d) Comment on how well the model's output agrees with the given data values,

(e) After how many years does the model predict that the world's population will first reach 3 billion?

(f) Compare your answers with those obtained by the discrete model in question two of the resource material for chapter $B1$.

Answers

Question 1

As $\frac{dy}{dx} = x^2(1+x^3)^4$ is of the form $\frac{dy}{dx} = f(x)$, the general solution is $y = \int x^2(1+x^3)^4\,dx + c$

If we re-write $\int x^2(1+x^3)^4\,dx$ as $\frac{1}{3}\int 3x^2(1+x^3)^4\,dx$, we can easily evaluate this integral because it has the form $\int \big(f(x)\big)^n f'(x)\,dx$, where $f(x) = 1+x^3$ and $n = 4$.

Hence, by result (2.3) on page 23 of Chapter $C2$,

$$\int x^2(1+x^3)^4\,dx = \tfrac{1}{3}\int 3x^2(1+x^3)^4\,dx = \tfrac{1}{3}\cdot\tfrac{1}{5}(1+x^3)^5 + c = \tfrac{1}{15}(1+x^3)^5 + c$$

Hence the general solution of the differential equation is $y = \frac{1}{15}(1+x^3)^5 + c$

(i) As $y(0) = 1$, we know that $y = 1$ when $x = 0$.

$\therefore c = \frac{14}{15}$, and so the particular solution for which $y(0) = 1$ is $y = \frac{1}{15}(1+x^3)^5 + \frac{14}{15}$

(ii) $y(-1) = \frac{1}{15}\left[1+(-1)^3\right]^5 + \frac{14}{15} = \frac{14}{15}$.

Question 2

As $\frac{dx}{dt} = e^{-t}$ is of the form $\frac{dx}{dt} = f(t)$, the general solution is $x = \int f(t)\,dt + c$.

$\therefore x = \int e^{-t}dt + c = -e^{-t} + c$

Now we have been given that $x = 3$ when $t = \ln 2$.

$\therefore 3 = -e^{-\ln 2} + c \Rightarrow c = 3 + e^{-\ln 2}$

Hence $c = 3 + e^{\ln 2^{-1}} = 3 + \frac{1}{2} = \frac{7}{2}$

$\therefore x = \frac{7}{2} - e^{-t}$

Question 3

With $y = 4e^{3x} - 3e^{4x} + xe^{4x}$, we have

$$\frac{dy}{dx} = 4\cdot 3e^{3x} - 3\cdot 4e^{4x} + e^{4x}\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(e^{4x}) = 12e^{3x} - 12e^{4x} + e^{4x} + 4xe^{4x}$$

$$= 12e^{3x} - 11e^{4x} + 4xe^{4x}$$

$$\frac{d^2y}{dx^2} = 12 \cdot 3e^{3x} - 11 \cdot 4e^{4x} + 4\left(e^{4x} + 4xe^{4x}\right)$$

$$= 36e^{3x} - 44e^{4x} + 4e^{4x} + 16xe^{4x}$$

$$= 36e^{3x} - 40e^{4x} + 16xe^{4x}$$

Substituting into the left hand side of the given differential equation, we have

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y$$

$$= 36e^{3x} - 40e^{4x} + 16xe^{4x} - 7\left(12e^{3x} - 11e^{4x} + 4xe^{4x}\right) + 12\left(4e^{3x} - 3e^{4x} + xe^{4x}\right)$$

$$= 36e^{3x} - 40e^{4x} + 16xe^{4x} - 84e^{3x} + 77e^{4x} - 28xe^{4x} + 48e^{3x} - 36e^{4x} + 12xe^{4x}$$

$$= e^{4x} \qquad \text{(as required)}$$

It remains to show that the given initial conditions are also satisfied. We have

$$y(0) = 4e^0 - 3e^0 + 0 = 4 - 3 = 1$$

$$y'(0) = 12e^0 - 11e^0 + 0 = 12 - 11 = 1$$

Hence $y = 4e^{3x} - 3e^{4x} + xe^{4x}$ satisfies $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{4x}, \quad y(0) = y'(0) = 1.$

Question 4

Using result (2.3) on page 16 of Chapter C3 gives

$$\frac{d}{dy}\left(y^2 - 5y\right)\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 8x\right)$$

$$\therefore \left(2y - 5\right)\frac{dy}{dx} = 3x^2 + 8$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 + 8}{2y - 5} \qquad \text{(as required)}$$

Question 5

(a) In order to solve $\frac{2y}{3x^2}\frac{dy}{dx}=\frac{y^2+1}{x^3-1}$ we must firstly re-write it as $\frac{2y}{y^2+1}\frac{dy}{dx}=\frac{3x^2}{x^3-1}$

If we then integrate both sides with respect to *x*, we have

$$\int\frac{2y}{y^2+1}dy=\int\frac{3x^2}{x^3-1}dx$$

Now both the integrals in this equation are of the form $\int\frac{f'(x)}{f(x)}dx$, and we know from equation (2.4) on page 23 of Chapter *C*2 that this evaluates to $\ln(f(x))$.

Hence the general solution is given by $\ln\left(y^2+1\right)=\ln\left(x^3-1\right)+c$.

Now we have been given that $y=1$ when $x=3$. Hence

$$c=\ln 2-\ln 26=\ln\left(\frac{2}{26}\right)=\ln\left(\frac{1}{13}\right)=\ln 13^{-1}=-\ln 13$$

Substituting back into the general solution gives $\ln\left(y^2+1\right)=\ln\left(x^3-1\right)-\ln 13$

$$\therefore \ln\left(y^2+1\right)=\ln\left(\frac{x^3-1}{13}\right)$$

$\therefore y^2+1=\frac{x^3-1}{13}$, which simplifies to $y^2=\frac{1}{13}\left(x^3-14\right)$

(b) In order to solve $\sec x\frac{dy}{dx}=e^{-y}$ we must firstly re-write it as $\frac{1}{e^{-y}}\frac{dy}{dx}=\frac{1}{\sec x}$

If we then integrate both sides with respect to *x*, we have

$\int\frac{1}{e^{-y}}dy=\int\frac{1}{\sec x}dx$, or equivalently $\int e^y dy=\int\cos x\,dx$. Hence $e^y=\sin x+c$.

Now we have been given that $y=\ln 2$ when $x=\frac{\pi}{6}$. Hence

$$\therefore e^{\ln 2}=\sin\tfrac{\pi}{6}+c\Rightarrow c=2-\tfrac{1}{2}=\tfrac{3}{2}$$

Hence the particular solution of $\sec x\frac{dy}{dx}=e^{-y}$ for which $y=\ln 2$ when $x=\frac{\pi}{6}$ is

$$e^y=\sin x+\tfrac{3}{2}\Rightarrow y=\ln\left(\sin x+\tfrac{3}{2}\right).$$

(c) To solve $(1+\cos 2x)\frac{dy}{dx} - e^y \sin 2x = 0$ we must firstly re-write it as $\frac{1}{e^y}\frac{dy}{dx} = \frac{\sin 2x}{1+\cos 2x}$.

If we then integrate both sides with respect to *x*, we have $\int \frac{1}{e^y}dy = \int \frac{\sin 2x}{1+\cos 2x}dx$.

Now $\int \frac{1}{e^y}dy = \int e^{-y}dy = -e^{-y}$

If we re-write $\int \frac{\sin 2x}{1+\cos 2x}dx$ as $-\frac{1}{2}\int \frac{-2\sin 2x}{1+\cos 2x}dx$, then $\int \frac{-2\sin 2x}{1+\cos 2x}dx$ is of the form $\int \frac{f'(x)}{f(x)}dx$, and we know from equation (2.4) on page 23 of Chapter *C*2 that this evaluates to $\ln(f(x))$.

$$\therefore \int \frac{\sin 2x}{1+\cos 2x}dx = -\frac{1}{2}\int \frac{-2\sin 2x}{1+\cos 2x}dx = -\frac{1}{2}\ln\left(1+\cos 2x\right)+c$$

$\therefore -e^{-y} = -\frac{1}{2}\ln\left(1+\cos 2x\right)+c$, or equivalently $e^{-y} = \frac{1}{2}\ln\left(1+\cos 2x\right)+c_1$

Now we have been given that $y = \ln 2$ when $x = \frac{\pi}{4}$. Hence

$$e^{-\ln 2} = \tfrac{1}{2}\ln 1 + c_1 \Rightarrow c_1 = \tfrac{1}{2}$$

Hence the particular solution of $(1+\cos 2x)\frac{dy}{dx} - e^y \sin 2x = 0$ for which $y = \ln 2$ when $x = \frac{\pi}{4}$ is $e^{-y} = \frac{1}{2}\ln\left(1+\cos 2x\right)+\frac{1}{2}$, or if you prefer $y = -\ln\left[\frac{1}{2}\left(\ln\left(1+\cos 2x\right)+1\right)\right]$

(d) To solve $\frac{1+y}{y}\frac{dy}{dx} - x = xy$ we must firstly separate the variables. We have

$$\frac{1+y}{y}\frac{dy}{dx} = x + xy \Rightarrow \frac{1+y}{y}\frac{dy}{dx} = x(1+y)$$

Hence $\int \frac{1}{y}dy = \int x dx \Rightarrow \ln y = \frac{1}{2}x^2 + c$

Now $y(1) = 1 \Rightarrow 0 = \frac{1}{2} + c$, from which $c = -\frac{1}{2}$.

$$\therefore \ln y = \frac{1}{2}x^2 - \frac{1}{2} \Rightarrow y = e^{\frac{1}{2}\left(x^2-1\right)}$$

(e) To solve $y\sin x\dfrac{dy}{dx}=\cos x$ we must firstly re-write it as $y\dfrac{dy}{dx}=\dfrac{\cos x}{\sin x}$

$$\therefore \int y\,dy=\int\frac{\cos x}{\sin x}dx$$

Now the integral $\int\dfrac{\cos x}{\sin x}dx$ is of the form $\int\dfrac{f'(x)}{f(x)}dx$, and we know from equation (2.4) on page 23 of Chapter $C2$ that this evaluates to $\ln(f(x))$.

Hence the general solution is given by $\dfrac{1}{2}y^2=\ln(\sin x)+c$

Now we have been given that $y=1$ when $x=\frac{\pi}{6}$.

$$\therefore c=\tfrac{1}{2}-\ln\left(\tfrac{1}{2}\right)=\tfrac{1}{2}-\ln 2^{-1}=\tfrac{1}{2}+\ln 2$$

Hence the particular solution for which $y\left(\frac{\pi}{6}\right)=1$ is given by $\frac{1}{2}y^2=\ln(\sin x)+\frac{1}{2}+\ln 2$.

We can simplify this solution further by multiplying through by 2 and then applying the laws of logarithms. This gives

$$y^2=2\ln(\sin x)+1+2\ln 2$$

$$\therefore y^2=\ln(\sin x)^2+\ln e+\ln 2^2$$

$$\therefore y^2=\ln(\sin^2 x)+\ln e+\ln 4$$

$$\therefore y^2=\ln\left[4e\sin^2 x\right]$$

(f) To solve $r\dfrac{d\psi}{dr}=\cos^2\psi$ we must firstly re-write it as $\dfrac{1}{\cos^2\psi}\dfrac{d\psi}{dr}=\dfrac{1}{r}$

Integrating both sides with respect to r gives $\displaystyle\int\frac{1}{\cos^2\psi}\,d\psi=\int\frac{1}{r}\,dr$

$$\therefore \int\sec^2\psi\,d\psi=\int\frac{1}{r}\,dr$$

Hence the general solution is given by $\tan\psi=\ln r+c$

Now we have been given that $\psi=\frac{\pi}{4}$ when $r=1$. Hence $c=1$.

$\therefore \tan\psi=\ln r+1$, from which $\psi=\arctan\left(\ln r+1\right)$

Question 6

(a) $\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)=\frac{1}{2}\left[\frac{x+1-(x-1)}{(x-1)(x+1)}\right]=\frac{1}{2}\left[\frac{2}{x^2-1}\right]=\frac{1}{x^2-1}$

(b) We have been given that $\frac{dy}{dx}=\frac{y+1}{x^2-1}$, and that $y=-5$ when $x=2$.

Separating the variables gives $\int\frac{1}{y+1}dy=\int\frac{1}{x^2-1}dx$.

Hence, by (a), we can write this as $\int\frac{1}{y+1}\,dy=\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

Integrating gives $\ln(y+1)=\frac{1}{2}\left[\ln(x-1)-\ln(x+1)\right]+c$

Applying the initial condition $y=5$ when $x=2$ gives $c=\frac{1}{2}\ln 3+\ln 6$

$\ln(y+1)=\frac{1}{2}\left[\ln(x-1)-\ln(x+1)\right]+\frac{1}{2}\ln 3+\ln 6$

$\therefore \ln(y+1)=\frac{1}{2}\ln(x-1)+\frac{1}{2}\ln 3+\ln 6-\frac{1}{2}\ln(x+1)$

$=\frac{1}{2}\ln(x-1)+\frac{1}{2}\ln 3+\frac{1}{2}\ln 6^2-\frac{1}{2}\ln(x+1)$

$=\frac{1}{2}\left[\ln\frac{3\times 36\times(x-1)}{x+1}\right]$

$=\ln\left(\frac{108(x-1)}{x+1}\right)^{1/2}$

$\therefore y+1=\left(\frac{108(x-1)}{x+1}\right)^{1/2}$, or equivalently $y=\sqrt{\frac{108(x-1)}{x+1}}-1$

Question 7

(a) We have been given that $t=5570$ when $m=\frac{1}{2}m_0$.

$\therefore$ Substituting into $m=m_0e^{-kt}$ gives $\frac{1}{2}m_0=m_0e^{-5570k}$

$\therefore e^{-5570k}=\frac{1}{2}\Rightarrow k=-\frac{1}{5570}\ln\frac{1}{2}\approx 1.244\times 10^{-4}$

(b) Substituting this value of k back into $m = m_0 e^{-kt}$ gives $m \approx m_0 e^{-0.0001244t}$

Hence, when $m \approx 0.53 m_0$, we have $0.53 \approx e^{-0.0001244t}$

$\therefore t \approx \dfrac{\ln 0.53}{-0.0001244}$, which we would sensibly round to about 5000 years.

Hence the iceman dates back to around 3000 BC.

Question 8

(a) Let the (constant) sum of the masses be M. Then $x + y = M \Rightarrow y = M - x$

$\therefore \dfrac{dx}{dt} \, \alpha \, xy \Rightarrow \dfrac{dx}{dt} = kxy$ (where k is the constant of proportionality)

$$\Rightarrow \frac{dx}{dt} = kx(M - x)$$

If we now put $a = M$, then $a = x + y$ is the sum of the masses and $\dfrac{dx}{dt} = kx(a - x)$.

(b) $\dfrac{1}{a}\left(\dfrac{1}{x} + \dfrac{1}{a-x}\right) = \dfrac{1}{a}\left(\dfrac{a - x + x}{x(a-x)}\right) = \dfrac{1}{a}\left(\dfrac{a}{x(a-x)}\right) = \left(\dfrac{1}{x(a-x)}\right)$

(c) Separating the variables gives $\displaystyle\int \frac{1}{x(a-x)}\,dx = k\int dt$

Hence, by (b), $\displaystyle\frac{1}{a}\int\left(\frac{1}{x} + \frac{1}{a-x}\right)dx = k\int dt$

$$\therefore \frac{1}{a}\left[\ln x - \ln(a - x)\right] = kt + c$$

$\therefore \ln x - \ln(a - x) = akt + c_1$ (where $c_1 = ac$ is an arbitrary constant of integration)

$$\therefore \ln\left(\frac{x}{a-x}\right) = akt + c_1$$

$\therefore \dfrac{x}{a-x} = Ae^{akt}$ (where $A = e^{c_1}$ is an arbitrary constant of integration)

Now we have been given that $x = \frac{1}{10}a$ when $t = 0$. Hence $A = \dfrac{\frac{1}{10}a}{a - \frac{1}{10}a} = \frac{1}{9}$.

$\therefore \dfrac{x}{a-x} = \dfrac{1}{9}e^{akt}$

If we now replace x with $a-y$, this equation becomes $\dfrac{a-y}{y} = \dfrac{1}{9}e^{akt}$

$\therefore$ when $y = \frac{1}{10}a$, we have $\dfrac{a-\frac{1}{10}a}{\frac{1}{10}a} = \dfrac{1}{9}e^{akt}$

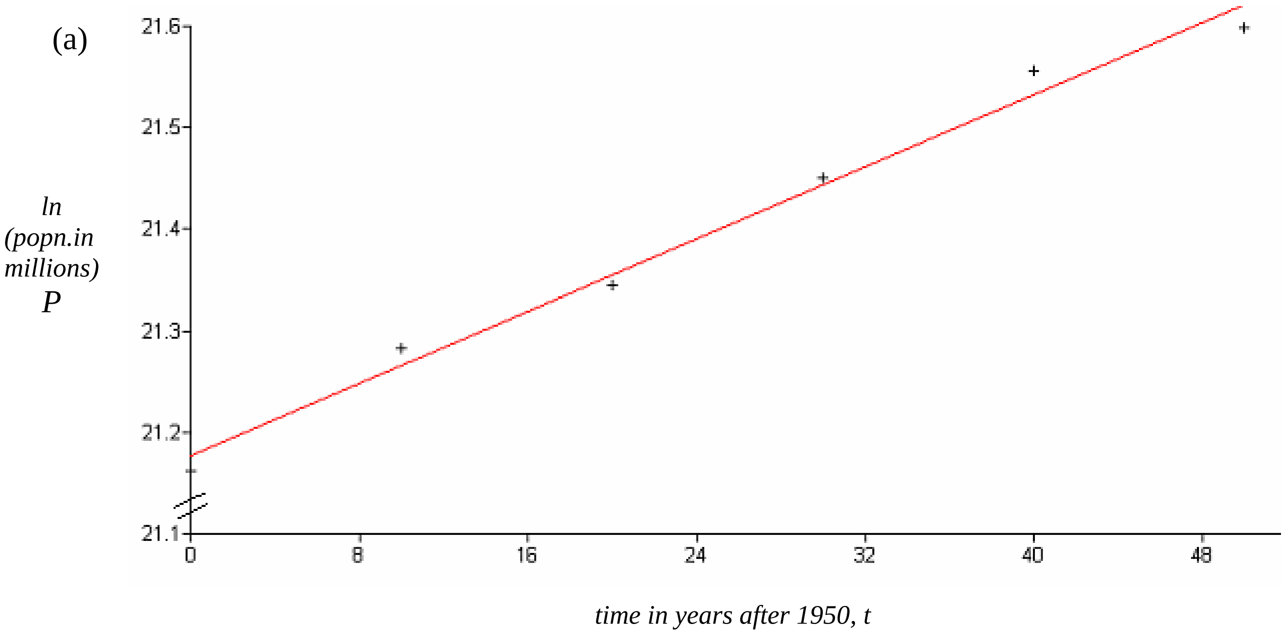
$\therefore \dfrac{1}{9}e^{akt} = 9 \Rightarrow e^{akt} = 81$, from which $akt = \ln 81$.

$\therefore t = \dfrac{1}{ak}\ln 81$

$\therefore t = \dfrac{2}{ak}\ln 9$ $\quad \left[\text{As } \ln 81 = \ln 9^2 = 2\ln 9\right]$

Question 9

(a)



$P = P_0 e^{Kt} \Rightarrow \ln P = Kt + \ln P_0$

As the points on the scatter diagram lie approximately on a straight line, it follows that a log-linear plot is appropriate for this set of data. Hence plotting values of $\ln P$ against t should give an approximate straight line of gradient K and intercept $\ln P_0$.

(b) K = gradient of line $\approx \dfrac{\ln\left(2.4\times10^3\right)-\ln\left(1.55\times10^3\right)}{50} \approx 0.009$.

intercept $= 21.18 \Rightarrow P_0 = e^{21.18} \approx 1579$ million.

(c) Hence the model predicts that $P \approx 1579e^{0.009t}$, where P is the population in millions and t is the time in years after 1950.

(d) To complete the table we use the exact value of $\dfrac{\ln\left(2.4\times10^3\right)-\ln\left(1.55\times10^3\right)}{50}=0.00874427\ldots$ calculated for K, rather than the rounded value of 0.009.

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400
Model's prediction	1579	1723	1881	2053	2240	2445
Percentage error	1.9	1.5	1.1	0.8	2.6	1.9

(e) As P is the population in millions, we need to solve $1579e^{0.009t}=3000$

$$\therefore 1579e^{0.009t}=3000 \Rightarrow e^{0.009t}\approx 1.9$$

$$\Rightarrow 0.008744\ldots t \approx \ln 1.9$$

$$\Rightarrow t \approx 73 \text{ years}$$

(f) The discrete model's predictions are shown in the table below.

Lower bounds for the world's population from 1900 to 1950						
Year	1900	1910	1920	1930	1940	1950
Year number	0	10	20	30	40	50
Lower bound (in millions)	1550	1750	1860	2070	2300	2400
Model's prediction	1550	1690	1850	2020	2200	2400
Percentage error	0	3.4	0.5	2.4	4.3	0

This shows that the discrete model approximated the data better at the start and the end of the fifty year period (which is not surprising if you recall how it was constructed), but that the continuous model was generally more accurate for the interim values.

Both models give similar predictions for the time at which the world's population will first reach 3 billion.