MST 121: Supplementary Resource Material for Chapter A3, Functions

1. For each of the following inequalities, write down the corresponding interval and describe it as open, half-open or closed.

   (a) $x \le 0$ (b) $2 < x < 5$ (c) $-2 \le x < -1$ (d) $x > 0$ (e) $-1 < x \le 1$

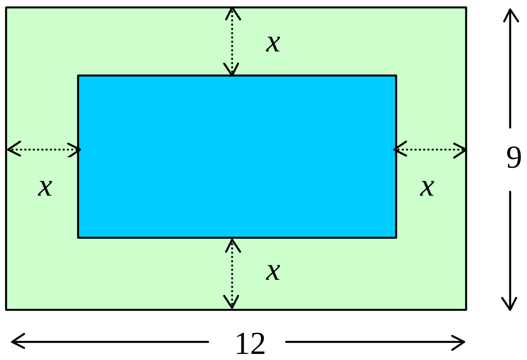
2. Draw the graphs of each of the following functions and, for each one, write down the largest possible domain, the corresponding image set and a suitable codomain:

   (a) $f(x) = x^2 - 6x + 11$ (b) $f(x) = x^3 - 5x^2 + 6x$ (c) $f(x) = \sin x$

   (d) $f(x) = \tan x$ (e) $f(x) = \frac{3}{x}$ (f) $f(x) = e^x$ (g) $f(x) = \ln x$ (h) $f(x) = \sqrt{x}$

   (i) $f(x) = \dfrac{3x+2}{x-1}$ (j) $f(x) = |2x - 3|$ (k) $f(x) = |\sin x|$ (l) $f(x) = |2 - x^2|$

3. A swimming pool is to be constructed in a country garden, 12 metres long by 9 metres wide. There is to be an area of lawn bordering the pool, for anyone who wishes to sunbathe, and the owner has specified that the pool and the border should occupy equal areas



   (a) Write down the largest *closed* interval consisting of values of x which are valid for this problem.
   (b) Write down expressions for the length and width of the pool, for those values of x identified in (a).
   (c) Use your answers to (a) and (b) to specify a function that returns the area of the pool.
   (d) Hence find the value of x that generates equal areas to the border and the pool.

4. (a) Express $f(x) = 2x^2 + 10x - 1$ in completed square form.
   (b) Deduce the sequence of transformations which transform the graph of $g(x) = x^2$ into that of $f(x)$.

5. Specify the sequence of transformations which transform the graph of $g(x) = \dfrac{1}{x}$ into that of $f(x) = \dfrac{x-3}{1+x}$. Hence sketch the graph of $f(x)$, indicating on your sketch the co-ordinates of any points where the graph intersects the co-ordinate axes. Show also on your sketch the co-ordinates of the point of intersection of the horizontal and vertical asymptotes.

6. Specify the sequence of transformations which transform the graph of $g(x) = |x|$ into that of $f(x) = |2x-3|$. Hence sketch the graph of $f(x)$, indicating on your sketch the co-ordinates of any points where the graph intersects the co-ordinate axes.

7. Superimpose the graph of $g(x) = |x+2|$ onto the graph of $f(x) = |2x-3|$ drawn in question six. Hence solve the equation $|2x-3| = |x+2|$.

8. Specify the sequence of transformations which transform the graph of $g(x) = \sin x$ into that of $f(x) = 5 + 2\sin(x - \frac{\pi}{4})$. Hence sketch the graph of $f(x)$, for $0 \le x \le 2\pi$.

9. (a) Sketch the graph of $f(x) = 3^x$ for $-1 \le x \le 1$

   (b) Explain why $\left(\frac{1}{3}\right)^x \equiv 3^{-x}$, and hence describe a single transformation which transforms the graph of $f(x) = 3^x$ into that of $g(x) = \left(\frac{1}{3}\right)^x$

   (c) Hence sketch the graph of $g(x)$ on the same set of axes used to sketch $f(x)$.

10. By considering the effect of an appropriate translation on the graph of the function $f(x) = e^x$, sketch the graph of the function $g(x) = e^{x-1}$.

11. For each of the following functions, state whether the function is increasing, decreasing, neither increasing nor decreasing, one-one or many-one:

    (a) $f(x) = \sqrt{x}$, (b) $f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ (c) $f(x) = 1 - x^2$ (d) $f(x) = |2x+1|$

12. (a) Sketch the graph of $f(x) = x^2$, $g(x) = \sqrt{x}$ and $h(x) = x$ on the same set of axes, taking the domain as $\{x \in \Re : x \ge 0\}$.

    (b) How are the graphs of $f(x)$ and $g(x)$ related to the graph of $h(x)$?

    (c) Suppose that you wished to solve the equation $f(x) = f^{-1}(x)$. Explain how you could do this by using $h(x)$, and without explicitly calculating the inverse function.

    (d) Solve the equation $f(x) = f^{-1}(x)$.

13. For each of (a)-(e) below, write down an expression for the inverse function and its corresponding domain.

    (a) $f(x) = x^2 - 12x + 9,\ 6 \le x \le 14$ (b) $f(x) = \ln(1+2x),\ x > -\frac{1}{2}$

    (c) $f(x) = \dfrac{3x-4}{2x+5},\ 0 \le x \le 10$ (d) $f(x) = 4x - 5,\ 8 \le x \le 15$ (e) $f(x) = e^{2x},\ x \in \Re$

14. The Fahrenheit temperature of a body is obtained by multiplying the Celsius temperature by $\frac{9}{5}$ and adding 32, the rule being valid for all Celsius temperatures greater than or equal to -273.

(a) Find a function to enable the Fahrenheit temperature to be calculated from the Celsius temperature.
(b) Find the inverse function which converts a Fahrenheit temperature to a Celsius temperature.

15. Without using a calculator, write down the exact values of each of the following expressions:

(a) $\log_{10} 100000$ (b) $\log_4 8$ (c) $\log_3(\frac{1}{27})$ (d) $\log_5 0.2$ (e) $\log_2(\frac{1}{16})$ (f) $\log_3 81$

16. Write each of the following equations in an alternative form, not involving logarithms:

(a) $\log_p q = r$ (b) $\log_{10} 100 = 2$ (c) $\log_t r = 1 - q$ (d) $\log_{2-x} y = z$

17. Evaluate each of the following expressions, giving your answers correct to three significant figures:

(a) $\log_7 11$ (b) $\log_{\frac{1}{3}} 8$ (c) $\log_{24} 27$ (d) $\log_2 1111$ (e) $\log_{132} 419$ (f) $\log_9 206$

18. Express each of the following in terms of $\log_a x$, $\log_a y$ and $\log_a z$:

(a) $\log_a\left(\dfrac{x^2y^5}{z^3}\right)$ (b) $\log_a xy^3z^5$ (c) $\log_a \sqrt{x^4y^2z^{2n}}$ (d) $\log_a\left(\dfrac{x^4y^8}{\sqrt{z^6}}\right)$

19. Express each of the following as a single logarithm:

(a) $\ln e^5 \times 2\log_b a$ (b) $3\log_a 4 + 5\log_a 2 - 6\log_a 4$ (c) $\frac{1}{2}\log_a 49 - \frac{1}{4}\log_a 81$

20. Solve each of the following equations, giving your answers as exact logarithms:

(a) $3^x = 5$ (b) $2^{3x-1} = 18$ (c) $5^{2x-1} = 2^{1-4x}$ (d) $6^{2-4x} = 7^{5x+3}$ (e) $\left(\dfrac{1}{5}\right)^{3x} = 6$

21. By firstly making an appropriate substitution or otherwise, find the exact values of x that satisfy each of the following equations:

(a) $2(3)^{2x} - 9(3)^x = -4$ (b) $3(4)^{2x} + 11(4)^x = 4$ (c) $25^x = 5^{x+1} - 6$

(d) $e^x - 6e^{-x} - 4 = 0$ (e) $5(2)^{2x} + 12(4)^x - 7 = 0$ (f) $2^x + 3(2)^{-x} - 6 = 0$

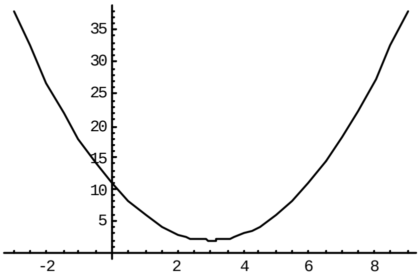
22. Write down the exact value, in radians, of each of the following expressions:

(a) $\arctan\left(-\dfrac{1}{\sqrt{3}}\right)$ (b) $\arccos\left(\dfrac{\sqrt{3}}{2}\right)$ (c) $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$ (d) $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$

Answers:

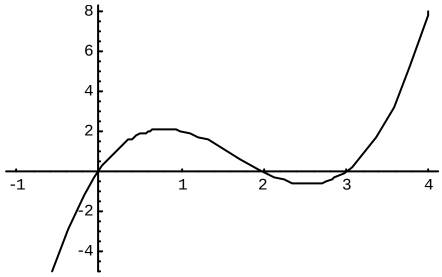
1. (a) $(-\infty, 0]$ closed (b) $(2,5)$ open (c) $[-2,-1)$ half open (d) $(0,\infty)$ open
(e) $(-1, 1]$ half open

2. (a)



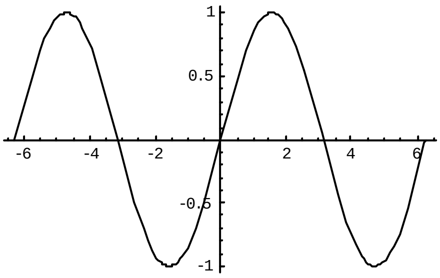
Domain = $\Re$
Image Set is $[2,\infty)$
Codomain = $\Re$

(b)



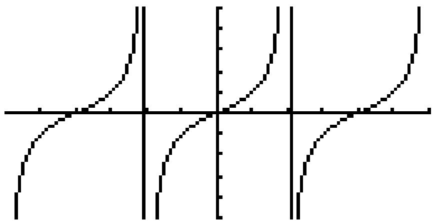
Domain = $\Re$
Image Set is $\Re$, or $(-\infty,\infty)$
Codomain = $\Re$

(c)



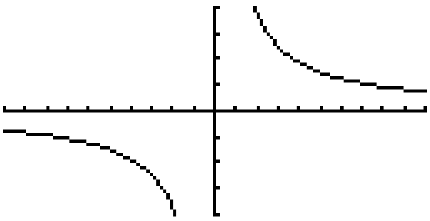
Domain = $\Re$
Image Set is $[-1,1]$
Codomain = $\Re$

(d)



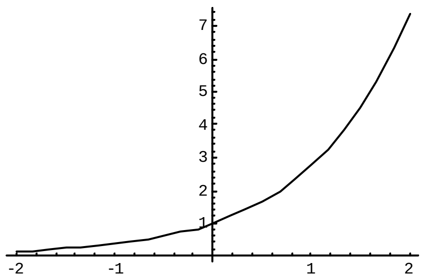
Domain = $\{x \in \Re : x \neq \frac{\pi}{2}(2n+1), n \in Z\}$
Image Set is $\Re$
Codomain = $\Re$

(e)



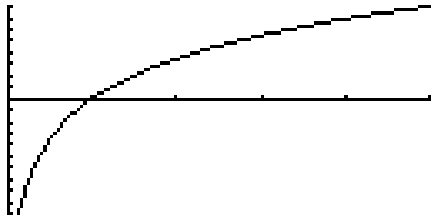
Domain = $\{x \in \Re : x \neq 0\}$
Image Set is $(-\infty,0) \cup (0,\infty)$
Codomain = $\Re$

(f)



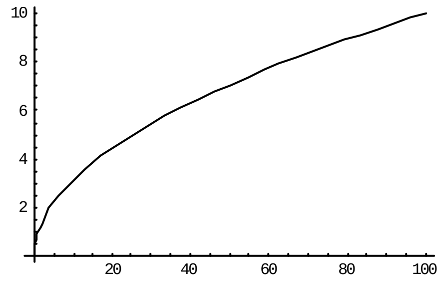
Domain = $\Re$
Image Set is $(0,\infty)$
Codomain = $\Re$

(g)



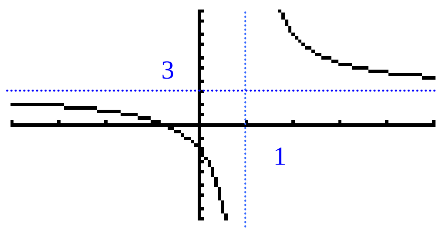
Domain = $(0,\infty)$
Image Set is $\Re$
Codomain = $\Re$

(h)



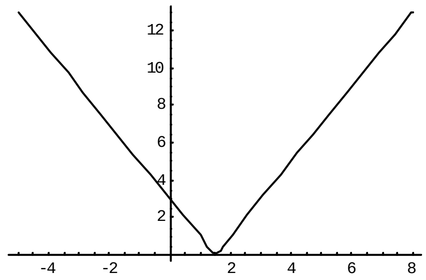
Domain = $[0,\infty)$
Image Set is $[0,\infty)$
Codomain = $\Re$

(i)



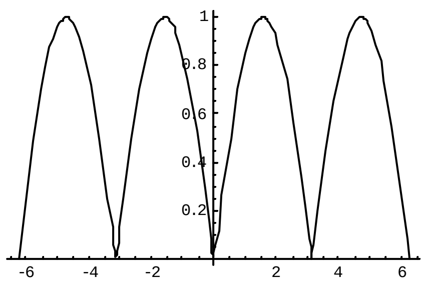
Domain = $(-\infty,1)\cup(1,\infty)$
Image Set is $(-\infty,3)\cup(3,\infty)$
Codomain = $\Re$

(j)



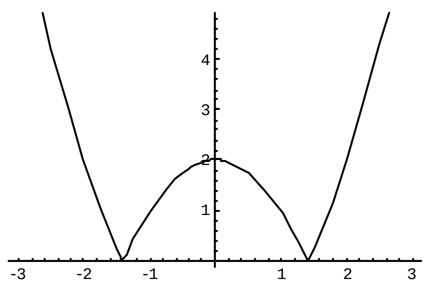
Domain = $\Re$
Image Set is $[0,\infty)$
Codomain = $\Re$

(k)



Domain = $\Re$
Image Set is $[0,1]$
Codomain = $\Re$

(l)



Domain = $\Re$
Image Set is $[0,\infty)$
Codomain = $\Re$

3. (a) $[0,\frac{9}{2}]$ (b) length $=12-2x$, width $=9-2x$

(c) $f(x)=(12-2x)(9-2x),\ x\in[0,\frac{9}{2}]$ (d) $(12-2x)(9-2x)=\frac{1}{2}\times 108 \Rightarrow x=1.5$

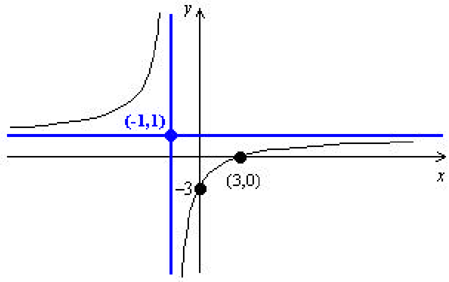
4. (a) $2\left(x+\frac{5}{2}\right)^2-\frac{27}{2}$

(b) ♦ Horizontal translation of $\frac{5}{2}$ units to the left, followed by

♦ y-scaling, with scale factor 2, followed by

♦ Vertical translation, by $\frac{27}{2}$ units downwards

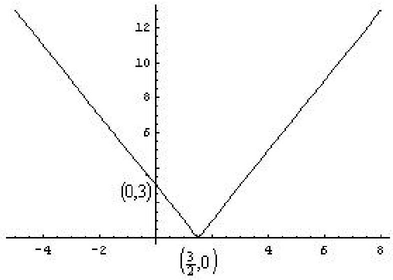
5. $\dfrac{x-3}{1+x}=\dfrac{1+x-4}{1+x}=\dfrac{1+x}{1+x}-\dfrac{4}{1+x}=1-\dfrac{4}{1+x}$



So the required transformations are

- ♦ y-scaling, scale factor -4, followed by
- ♦ Horizontal translation of 1 unit to the left, followed by
- ♦ Vertical translation, by 1 unit upwards

6. $f(x)=|2x-3|=2\left|x-\frac{3}{2}\right|$.

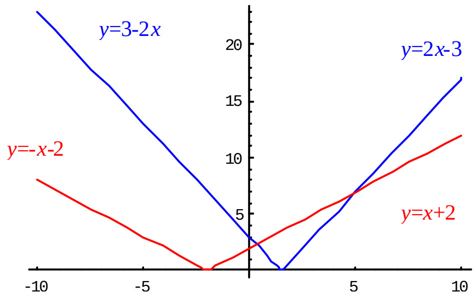


So the required transformations are

- ♦ Horizontal translation of $\frac{3}{2}$ unit to the right
- ♦ Followed by y-scaling, with scale factor 2

7. We need to solve

- ♦ $x+2=2x-3 \Rightarrow x=5$
- ♦ $x+2=3-2x \Rightarrow x=\frac{1}{3}$

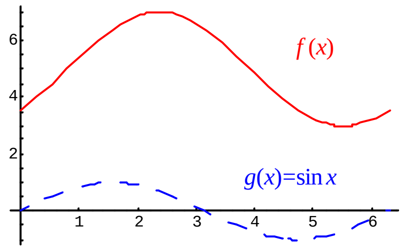


Hence there are two solutions to the equation $|2x-3|=|x+2|$, and these are $\underline{\underline{x=\frac{1}{3},5}}$

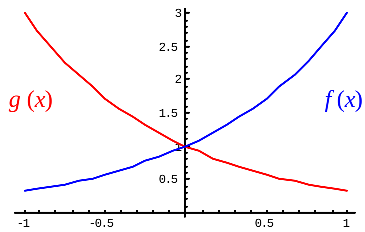
8. ♦ Horizontal translation of $\frac{\pi}{4}$ units to the right, followed by

♦ y-scaling, with scale factor 2, followed by

♦ Vertical translation, by 5 units upwards

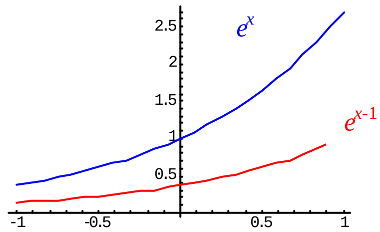


9. $\left(\dfrac{1}{3}\right)^x=\dfrac{1^x}{3^x}=\dfrac{1}{3^x}=3^{-x}$



Hence the graph of $\left(\frac{1}{3}\right)^x$ is obtained from that of 3^x by reflection in the y-axis.

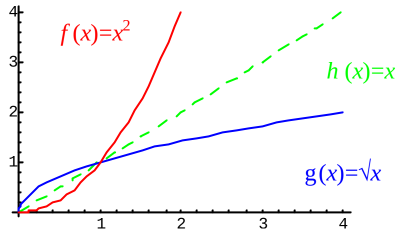
10. The graph of e^{x-1} is obtained from that of e^x by a horizontal translation of 1 unit to the right.



11. Increasing and one-one (b) Increasing and one-one (c) Neither increasing nor decreasing, and many-one (d) Neither increasing nor decreasing, and many-one

12. (b) $g(x)$ is a reflection of $f(x)$ in $h(x)$

(c) As $g(x)$ is a reflection of $f(x)$ in $y = x$, then $g(x) = f^{-1}(x)$. Hence any solutions of $f(x) = f^{-1}(x)$ are also solutions to $f(x) = x$.



(d) $\therefore\ f(x) = f^{-1}(x) \Rightarrow x^2 = x$, which has solutions $x = 0,1$

13. (a) $f^{-1}(x): x \mapsto 6 + \sqrt{x+27}, x \in [-27, 37]$ (b) $f^{-1}(x): x \mapsto \frac{1}{2}(e^x - 1), x \in \Re$

(c) $f^{-1}(x): x \mapsto \dfrac{4+5x}{3-2x}, x \in \left[-\dfrac{4}{5}, \dfrac{26}{25}\right]$ (d) $f^{-1}(x): x \mapsto \frac{1}{4}(x+5), x \in [27,55]$

(e) $f^{-1}(x): x \mapsto \frac{1}{2}\ln x, x \in (0, \infty)$

14. (a) $F(C) = \frac{9}{5}C + 32, C \in [-273, \infty)$ (b) $C^{-1}(F) = \frac{5}{9}(F - 32), F \in [-459.4, \infty)$

15. (a) 5 (b) $\frac{3}{2}$ (c) -3 (d) -1 (e) -4 (f) 4

16. (a) $p^r = q$ (b) $10^2 = 100$ (c) $t^{1-q} = r$ (d) $(2-x)^z = y$

17. (a) 1.23 (b) -1.89 (c) 1.04 (d) 10.1 (e) 1.24 (f) 2.42

18. (a) $2\log_a x + 5\log_a y - 3\log_a z$ (b) $\log_a x + 3\log_a y + 5\log_a z$

(c) $2\log_a x + \log_a y + n\log_a z$ (d) $4\log_a x + 8\log_a y - 3\log_a z$

19. (a) $\log_b a^{10}$ (b) $\log_a \frac{1}{2}$ (c) $\log_a\left(\frac{7}{3}\right)$

20. (a) $x = \dfrac{\ln 5}{\ln 3}$ (b) $x = \dfrac{2\ln 6}{3\ln 2}$ (c) $x = \dfrac{\ln 10}{2\ln 20}$ (d) $x = -\dfrac{\ln\left(\frac{343}{36}\right)}{\ln 21781872}$ (e) $x = -\dfrac{\ln 6}{3\ln 5}$

21. (a) $x = -\dfrac{\ln 2}{\ln 3}, \dfrac{2\ln 2}{\ln 3}$ (b) $x = -\dfrac{\ln 3}{2\ln 2}$ (c) $x = \dfrac{\ln 2}{\ln 5}, \dfrac{\ln 3}{\ln 5}$ (d) $x = \ln(2 + \sqrt{10})$

(e) $x = -\dfrac{\ln\left(\frac{17}{7}\right)}{2\ln 2}$ (f) $x = \dfrac{\ln(3 \pm \sqrt{6})}{\ln 2}$

22. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $-\frac{\pi}{3}$ (d) $\frac{\pi}{4}$