MS221: Tutorial Support Material for Chapter A3, Functions from geometry

1. Write down a parametrisation function for each of the following curves:

   (a) The circle $(x+4)^2+(y-2)^2=9$ (b) The parabola $y^2=20x$

   (c) The left branch of the hyperbola $\frac{1}{5}x^2-\frac{1}{12}y^2=1$

   In each case substitute your proposed parametrisation into the given Cartesian equation to confirm that your solution is correct.

2. This question concerns the quadratic curve L with equation $x^2-y^2-8x+10y=10$

   (a) Find the translation function that maps the quadratic curve L onto a conic K in standard position. Write down the equation of K.
   (b) Write down the translation function that maps K onto L
   (c) Find a parametrisation function (or functions) for the quadratic curve, L

3. Express each of the following translations in the form $t_{p,q}$

   (a) The inverse of $t_{-2,8}$ (b) the composite $t_{6,-1}\circ t_{2,-2}$

4. For each of the following angles, use the rotation formula

$$r_\theta : (x,y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

   to determine the function that specifies the rotation through that angle about the origin. In each case

   - Calculate the image of $(1,0)$ under the given function
   - Determine the inverse function
   - Check that the inverse function correctly maps the image of $(1,0)$ calculated above back onto $(1,0)$

   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{3\pi}{2}$

5. Use the result $r_\theta \circ r_\phi = r_{\theta+\phi}$, to determine each of the following composite functions. In each case, please ensure that your answer lies in the interval $(-\pi,\pi]$.

   (a) $r_{\frac{\pi}{3}}\circ r_{\frac{\pi}{6}}$ (b) $r_{\frac{\pi}{4}}\circ r_{-\frac{\pi}{6}}$ (c) $r_{\frac{2\pi}{3}}\circ r_{\frac{3\pi}{4}}$ (d) $r_{-\frac{\pi}{2}}\circ r_{-\frac{5\pi}{6}}$

6. Consider the reflection q and translation t with definitions given by

$$q:\Re^2\to\Re^2 \qquad\qquad t:\Re^2\to\Re^2$$
$$(x,y)\mapsto(-x,y) \quad \text{and} \quad (x,y)\mapsto(x,y-5)$$

   (a) Describe the geometric effects of the functions q and t.
   (b) What special name is given to the composite of these two functions
   (c) Determine the composite functions $q\circ t$ and $t\circ q$, and comment on the result.
   (d) Use the definitions of q and t to determine the image of (2,3), and verify that $q\circ t\,(2,3)=t\circ q\,(2,3)$.

7. Let q be the transformation that reflects the plane in the y-axis, and let r be the transformation that rotates the plane clockwise about the origin through $\frac{\pi}{6}$ radians. Algebraically, these have the forms

$$q : \Re^2 \to \Re^2 \qquad r : \Re^2 \to \Re^2$$
$$(x, y) \mapsto (-x, y) \qquad (x, y) \mapsto \left(\tfrac{1}{2}\left(\sqrt{3}x - y\right), \tfrac{1}{2}\left(x + \sqrt{3}y\right)\right)$$

(d) Determine the composite isometries $r \circ q$ and $q \circ r$.
(e) is it true that $r \circ q = q \circ r$?
(f) Use your answer to (b) to make a general statement about combinations of reflections and rotations.

8. Let l be the line through the origin that makes an angle θ radians with the direction of the positive x-axis. For each of the following values of θ, use the function

$$q_\theta : \Re^2 \to \Re^2$$
$$(x, y) \mapsto \left(x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta)\right)$$

to determine the image of an arbitrary point (x, y) under the reflection q_θ

(a) $\theta = \frac{\pi}{8}$ (b) $\theta = \frac{\pi}{12}$ (c) $\theta = \frac{\pi}{4}$ (d) $\theta = \frac{2\pi}{3}$

9. (a) Find $\sec\theta$ and $\cot\theta$, given that $\sin\theta = \frac{5}{13}$ and $\theta \in (\frac{\pi}{2}, \pi)$

(b) Find $\operatorname{cosec}\theta$ and $\cos\theta$, given that $\tan\theta = \frac{3}{4}$ and $\theta \in (\pi, \frac{3\pi}{2})$

(c) Find $\sin\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$, given that $\cot\theta = \frac{8}{15}$ and $\theta \in (0, \frac{\pi}{2})$

10. Use the addition formulas

$$\sin(\theta \pm \phi) = \sin\phi\cos\theta \pm \cos\phi\sin\theta$$
$$\cos(\theta \pm \phi) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$$

to find the exact values of

(a) $\sin\frac{5\pi}{12}$ (b) $\operatorname{cosec}\frac{5\pi}{12}$ (c) $\sec\frac{\pi}{12}$ (d) $\cos\frac{7\pi}{12}$

11. Use $\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$ to find the exact values of (a) $\tan\frac{5\pi}{12}$ (b) $\cot\frac{7\pi}{12}$

12. Use the half angle formulas $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$ to find the exact values of (a) $\sin\frac{3\pi}{8}$ and (b) $\cos\frac{3\pi}{8}$.

and hence deduce the exact values of (c) $\tan\frac{3\pi}{8}$ (d) $\operatorname{cosec}\frac{3\pi}{8}$ (e) $\sec\frac{3\pi}{8}$ (f) $\cot\frac{3\pi}{8}$

13. Use the half angle formulas to find $\sin\theta$ and $\cos\theta$, where θ is the angle in the interval $(0, \frac{\pi}{2})$ for which $\cos(2\theta) = -\frac{5}{9}$.

14. Use the techniques introduced in section 4 of the unit to sketch the curves with equations $14x^2 + 5xy + 2y^2 - 28 = 0$ and $5x^2 + 8xy - 10y^2 - 12 = 0$

Answers:

1. (a) This curve is a circle, centre $(-4,2)$ and radius 3. Hence

$$p:[0,2\pi]\to\Re^2$$
$$t\mapsto(3\cos t-4,\ 3\sin t+2)$$

(b) This curve is a parabola in standard position, $y^2=4ax$, with $a=5$. Hence

$$p:\Re\to\Re^2$$
$$t\mapsto(5t^2,10t)$$

(g) This curve is a hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ in standard position with $a=\sqrt{5}$ and $b=\sqrt{12}=2\sqrt{3}$. Using $(x,y)=(a\sec t, b\tan t)$ gives the parametrisation of the left branch of the hyperbola as

$$p:\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right)\to\Re^2$$
$$t\mapsto\left(\sqrt{5}\sec t,\ 2\sqrt{3}\tan t\right)$$

2. (a) $x^2-y^2-8x+10y=10\Rightarrow(x-4)^2-(y-5)^2=1$. So the transformation t is

$$t:\Re^2\to\Re^2$$
$$(x,y)\mapsto(x-4,y-5)$$

The image of L under t is the hyperbola K with equation $x^2-y^2=1$

(b) $t:\Re^2\to\Re^2$
$$(x,y)\mapsto(x+4,y+5)$$

(c) The rules are the same for each branch of the hyperbola, but the domains of the parameter t are not. The parametrisations of the left and right branches of the hyperbola are respectively

$$p:\left(\tfrac{\pi}{2},\tfrac{3\pi}{2}\right)\to\Re^2 \qquad t\mapsto\left(4+\sec t,\ 5+\tan t\right)$$

and

$$p:\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right)\to\Re^2 \qquad t\mapsto\left(4+\sec t,\ 5+\tan t\right)$$

3. (a) $t_{2,-8}$ (b) $t_{8,-3}$

4. (a)
- $r_{\frac{\pi}{2}}:\Re^2\to\Re^2$, $(x,y)\mapsto(-y,x)$
- $(1,0)\mapsto(0,1)$
- $r_{-\frac{\pi}{2}}:\Re^2\to\Re^2$, $(x,y)\mapsto(y,-x)$

(b)
- $r_{\frac{\pi}{3}}:\Re^2\to\Re^2$, $(x,y)\mapsto\left(\tfrac{1}{2}\left(x-\sqrt{3}y\right),\tfrac{1}{2}\left(\sqrt{3}x+y\right)\right)$
- $(1,0)\mapsto\left(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}\right)$
- $r_{-\frac{\pi}{3}}:\Re^2\to\Re^2$, $(x,y)\mapsto\left(\tfrac{1}{2}\left(x+\sqrt{3}y\right),\tfrac{1}{2}\left(y-\sqrt{3}x\right)\right)$

(c) ♦ $r_{\frac{\pi}{6}} : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(\frac{1}{2}\left(\sqrt{3}x - y\right), \frac{1}{2}\left(x + \sqrt{3}y\right)\right)$

♦ $(1,0) \mapsto \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

♦ $r_{-\frac{\pi}{6}} : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(\frac{1}{2}\left(\sqrt{3}x + y\right), \frac{1}{2}\left(\sqrt{3}y - x\right)\right)$

(d) The results here are the same as those obtained in (a), for a rotation through $\frac{\pi}{2}$ radians anticlockwise is equivalent to a rotation through $\frac{3\pi}{2}$ radians clockwise

♦ $r_{\frac{\pi}{2}} : \Re^2 \to \Re^2$
$(x, y) \mapsto (-y, x)$

♦ $(1,0) \mapsto (0,1)$

♦ $r_{-\frac{\pi}{2}} : \Re^2 \to \Re^2$
$(x, y) \mapsto (y, -x)$

5. (a) $r_{\frac{\pi}{2}}$ (b) $r_{\frac{\pi}{12}}$ (c) $r_{\frac{17\pi}{12}} = r_{-\frac{7\pi}{12}}$ (d) $r_{-\frac{4\pi}{3}} = r_{\frac{2\pi}{3}}$

6. (a) q describes reflection in the y-axis, and t describes a vertical translation through 5 units downwards.
(b) Glide reflection
(c) $t \circ q : \Re^2 \to \Re^2$
$(x, y) \mapsto (-x, y - 5)$
and
$q \circ t : \Re^2 \to \Re^2$
$(x, y) \mapsto (-x, y - 5)$

It does not matter in which order we form the composite, as $t \circ q = q \circ t$

(d) $(2,3) \mapsto (-2,-2)$

7. (a) $r \circ q : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(-\frac{1}{2}\left(\sqrt{3}x + y\right), \frac{1}{2}\left(\sqrt{3}y - x\right)\right)$,
$q \circ r : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(-\frac{1}{2}\left(\sqrt{3}x - y\right), \frac{1}{2}\left(x + \sqrt{3}y\right)\right)$

(b) No. $r \circ q \neq q \circ r$

(c) Composition of rotations and reflections is not commutative. In other words, it matters in which order the transformations are composed.

8. (a) $q_{\frac{\pi}{8}} : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y)\right)$

(b) $q_{\frac{\pi}{12}} : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(\frac{1}{2}\left(\sqrt{3}x + y\right), \frac{1}{2}\left(x - \sqrt{3}y\right)\right)$

(c) $q_{\frac{\pi}{4}} : \Re^2 \to \Re^2$
$(x, y) \mapsto (y, x)$

(d) $q_{\frac{2\pi}{3}} : \Re^2 \to \Re^2$
$(x, y) \mapsto \left(-\frac{1}{2}\left(x + \sqrt{3}y\right), \frac{1}{2}\left(y - \sqrt{3}x\right)\right)$

9. (a) $\sec\theta = -\frac{13}{12}$ and $\cot\theta = -\frac{12}{5}$ (b) $\operatorname{cosec}\theta = -\frac{5}{3}$ and $\cos\theta = -\frac{4}{5}$
(c) $\sin\theta = \frac{15}{17}$, $\sec\theta = \frac{17}{8}$ and $\operatorname{cosec}\theta = \frac{17}{15}$

10. (a) $\frac{1}{4}\left(\sqrt{2} + \sqrt{6}\right)$ (b) $\sqrt{6} - \sqrt{2}$ (c) $\sqrt{6} - \sqrt{2}$ (d) $\frac{1}{4}\left(\sqrt{2} - \sqrt{6}\right)$

11. (a) $2 + \sqrt{3}$ (b) $\sqrt{3} - 2$

12. (a) $\sin\left(\frac{3\pi}{8}\right) = \sqrt{\frac{1 - \cos\frac{3\pi}{4}}{2}} = \sqrt{\frac{1 - (-\frac{1}{\sqrt{2}})}{2}} = \frac{1}{2}\sqrt{2 + \sqrt{2}}$ (b) $\frac{1}{2}\sqrt{2 - \sqrt{2}}$

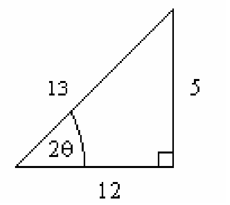
(c) $\dfrac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$ (d) $\dfrac{2}{\sqrt{2+\sqrt{2}}}$ (e) $\dfrac{2}{\sqrt{2-\sqrt{2}}}$ (f) $\dfrac{1}{1+\sqrt{2}} = 1+\sqrt{2}$

13. Since $\theta \in (0, \frac{\pi}{2})$, we know that $\sin\theta$ and $\cos\theta$ are both positive. Hence

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1+\frac{5}{9}}{2}} = \sqrt{\frac{7}{9}} = \frac{1}{3}\cdot\sqrt{7}$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1-\frac{5}{9}}{2}} = \sqrt{\frac{2}{9}} = \frac{1}{3}\cdot\sqrt{2}$$

14.
- $14x^2 + 5xy + 2y^2 - 28 = 0$ is of the general form $Ax^2 + Bxy + Cy^2 + F = 0$ where $A = 14, B = 5, C = 2$ and $F = -28$.
- As $A \neq C$, we have $\tan 2\theta = \dfrac{B}{A-C} = \dfrac{5}{12}$. Hence the angle of inclination, θ, is given by $\theta = \frac{1}{2}\arctan\frac{5}{12} \approx 11^{\circ}$
- We must now apply equations 4.6 on page 40 of Unit A3. Firstly, however, we note that $\tan 2\theta = \frac{5}{12} \Rightarrow \sin 2\theta = \frac{5}{13}$ and $\cos 2\theta = \frac{12}{13}$



$\therefore \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) = \frac{1}{2}\left(1+\frac{12}{13}\right) = \frac{25}{26}$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) = \frac{1}{2}\left(1-\frac{12}{13}\right) = \frac{1}{26}$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}\times\frac{5}{13} = \frac{5}{26}$

$\therefore A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = 14\times\frac{25}{26} + 5\times\frac{5}{26} + 2\times\frac{1}{26} = \frac{29}{2}$

$B' = 0$ (by construction, as our intention is to eliminate the term in xy)

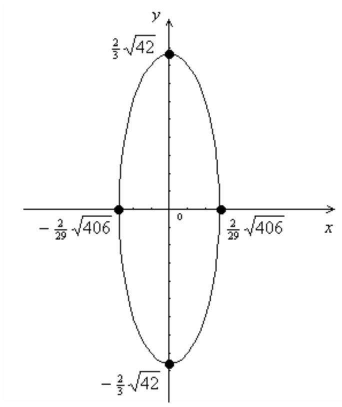
$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta = 14\times\frac{1}{26} - 5\times\frac{5}{26} + 2\times\frac{25}{26} = \frac{3}{2}$

$D' = E' = 0$, and $F' = F = -28$

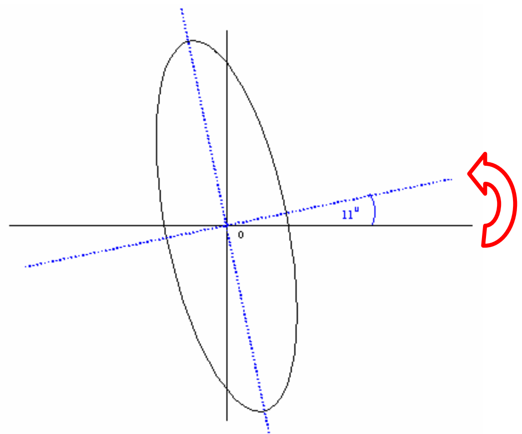
- Hence the equation of the conic in standard position is

$$A'x^2 + C'y^2 + F' = 0 \Rightarrow \frac{29}{2}x^2 + \frac{3}{2}y^2 - 28 = 0$$

This simplifies to $29x^2 + 3y^2 = 56$, which is an ellipse that we can easily sketch:



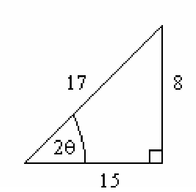
To sketch the given conic, we must rotate the ellipse on the left through 11° anticlockwise.



- $5x^2 + 8xy - 10y^2 - 12 = 0$ is of the general form $Ax^2 + Bxy + Cy^2 + F = 0$ where $A = 5, B = 8, C = -10$ and $F = -12$.

- As $A \neq C$, we have $\tan 2\theta = \dfrac{B}{A-C} = \dfrac{8}{15}$. Hence the angle of inclination, θ, is given by $\theta = \frac{1}{2}\arctan\frac{8}{15} \approx 14^{\circ}$

- We must now apply equations 4.6 on page 40 of Unit A3. Firstly, however, we note that $\tan 2\theta = \frac{8}{15} \Rightarrow \sin 2\theta = \frac{8}{17}$ and $\cos 2\theta = \frac{15}{17}$



$\therefore \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) = \frac{1}{2}\left(1+\frac{15}{17}\right) = \frac{16}{17}$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) = \frac{1}{2}\left(1-\frac{15}{17}\right) = \frac{1}{17}$

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}\times\frac{8}{17} = \frac{4}{17}$

$\therefore A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta = 5\times\frac{16}{17} + 8\times\frac{4}{17} - 10\times\frac{1}{17} = 6$

$B' = 0$ (by construction, as our intention is to eliminate the term in xy)

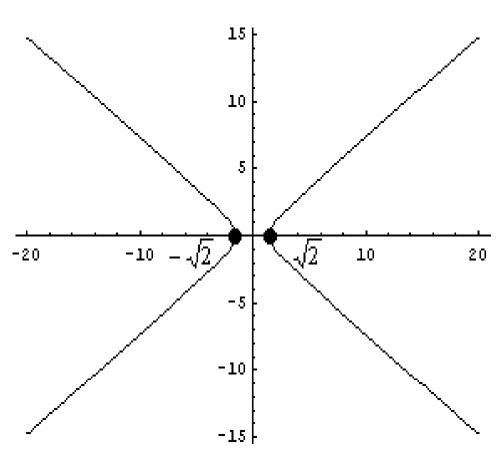
$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta = 5\times\frac{1}{17} - 8\times\frac{4}{17} - 10\times\frac{16}{17} = -11$

$D' = E' = 0$, and $F' = F = -12$

- Hence the equation of the conic in standard position is

$$A'x^2 + C'y^2 + F' = 0 \Rightarrow 6x^2 - 11y^2 - 12 = 0$$

This is a hyperbola in standard position, which we can easily sketch:



To sketch the given conic, we must rotate the one in standard position through 14° anticlockwise.